

Dimension formulas
for the spaces of modular
forms.

Notation: Let $D = \sum_x n_x \cdot x$ be a divisor on X
we write $D \geq 0$ if $n_x \geq 0$ for all $x \in X$.

Let D be any divisor on X . Consider the following vector space:

$$L(D) = \{f \in \mathbb{C}(X) \mid f \equiv 0 \text{ or } \operatorname{div}(f) + D \geq 0\}$$

$$\ell(D) := \dim L(D)$$

Theorem (Riemann-Roch) Let X be a compact Riemann surface of genus g . Let $\operatorname{div}(\gamma)$ be a canonical divisor on X . Then for any divisor $D \in \operatorname{Div}(X)$

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\gamma) - D)$$

Corollary: Let X be a compact Riemann surface of genus g .

Let $\text{div}(\lambda)$ be a canonical divisor on X .

Let D be any divisor on X

(a) $\ell(\text{div}(\lambda)) = g$

(b) $\deg(\text{div}(\lambda)) = 2g - 2$

(c) If $\deg(D) < 0$ then $\ell(D) = 0$

(d) If $\deg(D) > 2g - 2$ then $\ell(D) = \deg(D) - g + 1$

Proof:

(a) We apply the RR formula

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\lambda) - D)$$

to $D = 0$:

$$\ell(0) = \deg(0) - g + 1 + \ell(\operatorname{div}(\lambda)).$$

Note that $\mathcal{L}(0) = \mathbb{C}$ and $\ell(0) = 1$.

Thus $\ell(\operatorname{div}(\lambda)) = g$ 

(b) Apply RR to $D = \operatorname{div}(\lambda)$

$$\ell(\operatorname{div}(\lambda)) = \deg(\operatorname{div}(\lambda)) - g + 1 + \ell(0)$$

Now part (a) implies

$$\deg(\operatorname{div}(\lambda)) = 2g - 2$$
 

(c) Suppose that $f \in L(D) \setminus \{0\}$

Then $\text{div}(f) + D \geq 0 \Rightarrow \deg(\text{div}(f) + D) \geq 0$

On the other hand : $\deg(\text{div}(f)) = 0$, $\deg(D) < 0$

Hence $\deg(\text{div}(f) + D) < 0 \nRightarrow \Rightarrow \ell(D) = 0 \quad \square$

(d) Suppose that $\deg(D) > 2g - 2$

by RR :

$$\ell(D) = \deg(D) - g + 1 + \ell(\text{div}(\lambda) - D) \quad (*)$$

by part (b)

$$\deg(\text{div}(\lambda) - D) = 2g - 2 - \deg(D) < 0.$$

Therefore, by part (c) $\ell(\text{div}(\lambda) - D) = 0$

Hence (*) becomes : $\ell(D) = \deg(D) - g + 1 \quad \square$

Divisor of a modular form.

Let f be a modular form of weight k and group Γ .

Recall **Theorem**: Let $\pi: \mathfrak{h} \rightarrow X(\Gamma)$ be the natural projection map.

There exists a unique meromorphic differential ω of degree $k/2$ on $X(\Gamma)$

such that $\pi^*(\omega) = f(z)(dz)^{k/2}$

Abuse of notation: $\omega := f(z)(dz)^{k/2}$

What can we say about $\text{div}(\omega)$?

We consider 3 cases

- regular points
- elliptic points
- cusps

- Suppose that $P \in X(\Gamma)$ is a regular point, then

$$\text{ord}_P(\omega) = \text{ord}_{z_P}(f),$$

where $z_P \in \mathfrak{h}$, $\pi(z_P) = P$

$\pi: \mathfrak{h} \rightarrow X(\Gamma)$
is the natural
projection

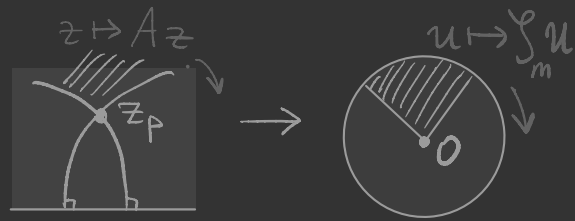
- Suppose that $P \in X(\Gamma)$ is an elliptic point of order m .

lemma: Let A be a generator of $\text{Stab}_{z_P}(\bar{\Gamma})$.

There exists a biholomorphic function $u: h \rightarrow D$ such that

$$\nabla u(z_P) = 0$$

$$\nabla u(Az) = e^{\frac{2\pi i}{m}} u(z) \quad =: \zeta_m$$



Then $q = u^m$ is a local parameter around P on $X(\Gamma)$.

$$f(z)(dz)^{k/2} = f(u)\left(\frac{dz}{du}\right)^{k/2} \cdot (du)^{k/2} =: \tilde{f}(u)(du)^{k/2}$$

We have: $\text{ord}_{z=z_P} f(z) = \text{ord}_{u=0} \tilde{f}(u) =: \eta_P$

$$\tilde{f}(\zeta_m u) \cdot (d\zeta_m u)^{k/2} = \tilde{f}(u) (du)^{k/2} \Rightarrow \tilde{f}(\zeta_m u) = \zeta_m^{-k/2} \tilde{f}(u)$$

The Laurent expansion of $\tilde{f}$ around $u=0$ has the following form:

$$\tilde{f}(u) = \sum_{\substack{n \equiv -k/2 \pmod{m} \\ n \geq n_p}} c_{\tilde{f}}(n) u^n$$

$$q = u^m \quad dq = m \cdot u^{m-1} du$$

$$\omega = \tilde{f}(u(q)) \left(\frac{du}{dq} \right)^{k/2} \cdot (dq)^{k/2} = \text{all powers } n + k/2 - m \cdot k/2 \text{ are divisible by } m$$

$$= \tilde{f}(u) m^{-k/2} u^{k/2 - m \cdot k/2} (dq)^{k/2}$$

Therefore, this function is a meromorphic function of $q = u^m$

$$= \sum_{\substack{n \equiv -k/2 \pmod{m} \\ n \geq n_p}} c_{\tilde{f}}(n) \cdot m^{-k/2} \cdot u^{n + k/2 - m \cdot k/2} (dq)^{k/2}$$

$$= \left(c_0 q^{\frac{n_p + k/2}{m} - k/2} + \dots \right) (dq)^{k/2}$$

$$\text{ord}_p \omega = \frac{n_p + k/2}{m} - k/2$$

Recall:

We assume that k is even

① Suppose that P is a cusp of $X(\Gamma)$.

Let $q = e^{2\pi iz/h}$ be a local parameter around P

$$f(z) = \sum_{n=n_p}^{\infty} c_f(n) e^{\frac{2\pi i n}{h} z}$$

$$\omega = f(z) (dz)^{k/2}$$

$$dq = \frac{2\pi i}{h} q \cdot dz$$

$$\omega(q) = f(q(z)) \left(\frac{dz}{dq} \right)^{k/2} (dq)^{k/2} = f(q) \left(\frac{h}{2\pi i} q^{-1} \right)^{k/2} (dq)^{k/2}$$

$$= f(q) q^{-k/2} \left(\frac{h}{2\pi i} \right)^{k/2} (dq)^{k/2}$$

$$\text{ord}_P \omega = n_p - k/2$$

Thus, we find:

$$\begin{aligned}\operatorname{div}(\omega) &= \sum_{P \text{ regular}} n_P \cdot P \\ &+ \sum_{P \text{ cusp}} (n_P - k/2) \cdot P \\ &+ \sum_{P \text{ elliptic}} \left(\frac{1}{m_P} \cdot n_P - \left(1 - \frac{1}{m_P}\right) \cdot \frac{k}{2} \right) P\end{aligned}$$

We introduce the following notations:

$$\operatorname{div}^*(dz) := \sum_{P \text{ cusp}} (-P) + \sum_{\substack{P \text{ elliptic} \\ \text{of order } 3}} \left(-\frac{2}{3}\right) \cdot P + \sum_{\substack{P \text{ elliptic} \\ \text{of order } 2}} \left(-\frac{1}{2}\right) \cdot P$$

$$\operatorname{div}^*(f) := \sum_{P \text{ regular}} n_P \cdot P + \sum_{P \text{ cusp}} n_P \cdot P + \sum_{P \text{ elliptic}} \frac{1}{m_P} \cdot n_P \cdot P$$

We have: $\operatorname{div}(\omega) = \operatorname{div}^*(f) + k/2 \cdot \operatorname{div}^*(dz)$

Now we can describe elements of $\mathcal{M}_k(\Gamma)$:

$$f \in \mathcal{M}_k(\Gamma) \Leftrightarrow \operatorname{div}^*(f) \geq 0 \Leftrightarrow \operatorname{div}(w) - k/2 \operatorname{div}^*(dz) \geq 0$$

Let f_0 be any element of $\mathcal{A}_k(\Gamma)$ (recall: $\mathcal{A}_k(\Gamma)$ is the space of meromorphic modular forms)

Then $w_0 := f_0(z)(dz)^{k/2}$ is a meromorphic differential.

We have: $\mathcal{A}_k(\Gamma) = \{ g \cdot f_0 \mid g \in \mathcal{A}_0(\Gamma) \cong \mathbb{C}(X(\Gamma)) \}$

$\mathcal{M}_k(\Gamma)$ is a subspace of $\mathcal{A}_k(\Gamma)$

$f_0 \cdot g \in \mathcal{M}_k(\Gamma)$ if and only if $n_p(f_0 \cdot g) \geq 0$
for all $p \in X(\Gamma)$.

This is equivalent to:

$$\operatorname{div}^*(f_0 \cdot g) \geq 0$$

Recall: $\operatorname{div}(w_0 \cdot g) = \operatorname{div}^*(f_0 \cdot g) + \frac{k}{2} \operatorname{div}^*(dz)$

Therefore, $f_0 \cdot g$ belongs to $M_k(\Gamma)$ if and only if

$$\operatorname{div}(w_0 \cdot g) - \frac{k}{2} \operatorname{div}^*(dz) \geq 0$$

Use Riemann-Roch?

$$\operatorname{div}(g) + \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \geq 0$$

We introduce the following notation:

for $v_i \in \mathbb{Q}$ $\lfloor \sum v_i P_i \rfloor := \sum \lfloor v_i \rfloor \cdot P_i$,

here $\lfloor v \rfloor$ is an integral part of a rational number v

By definition, this condition is equivalent to:

$$g \in \mathcal{L} \left(\lfloor \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \rfloor \right)$$

⚠ This is not a
divisor on $X(\Gamma)$

We have established that

$$\dim(\mathcal{M}_k(\Gamma)) = \ell\left(\lfloor \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \rfloor\right)$$

where w_0 is any element of $\Omega^{\otimes k/2}(X(\Gamma)) \setminus \{0\}$.

Now it suffices to compute $\ell\left(\lfloor \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \rfloor\right)$.

First, we show that

$$\deg\left(\lfloor \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \rfloor\right) > 2g - 2$$

By a corollary to RR

$$\deg(\operatorname{div}(w_0)) = \frac{k}{2}(2g-2)$$

Also we have:

$$\deg\left(\lfloor \operatorname{div}(w_0) - \frac{k}{2} \operatorname{div}^*(dz) \rfloor\right)$$

$$= \frac{k}{2}(2g-2) + \lfloor \frac{k}{4} \rfloor \varepsilon_2 + \lfloor \frac{k}{3} \rfloor \varepsilon_3 + \frac{k}{2} \varepsilon_\infty$$

$$\geq \frac{k}{2}(2g-2) + \frac{k-2}{4} \varepsilon_2 + \frac{k-2}{3} \varepsilon_3 + \frac{k}{2} \varepsilon_\infty$$

$$= (2g-2) + \frac{k-2}{2} \left(2g-2 + \frac{\varepsilon_2}{2} + \frac{2\varepsilon_3}{3} + \varepsilon_\infty \right) + \varepsilon_\infty$$

$$\geq 2g-2 + \varepsilon_\infty > 2g-2$$

By our
computation
of the
genus of $X(\Gamma)$

$$= \frac{d}{6} > 0$$

This proves the Claim.

Now the Riemann-Roch theorem
(+ the corollary) implies:

$$\begin{aligned} \dim \left(\mathcal{L} \left(\mathcal{L} \operatorname{div}(\omega_0) - \frac{k}{2} \operatorname{div}^*(dz) \right) \right) &= \\ &= \deg \left(\mathcal{L} \operatorname{div}(\omega_0) - \frac{k}{2} \operatorname{div}^*(dz) \right) - g(X(\Gamma)) + 1 \end{aligned}$$

Theorem: For even k

$$\dim (\mathcal{M}_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \varepsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2} \varepsilon_\infty & \text{if } k \geq 2 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$S_k(\Gamma) \subset M_k(\Gamma)$$

$$\dim(S_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \lfloor \frac{k}{4} \rfloor \varepsilon_2 + \lfloor \frac{k}{3} \rfloor \varepsilon_3 + (\frac{k}{2}-1)\varepsilon_\infty & \text{if } k \geq 4 \\ g & \text{if } k = 2 \\ 0 & \text{if } k \leq 0 \end{cases}$$

Exercise: Suppose that $f \in S_2(\Gamma)$ then $f(z)dz$ defines a holomorphic differential on $X(\Gamma)$.

Moreover, $\Omega_{\text{hol}}^1(X(\Gamma)) \simeq S_2(\Gamma)$.

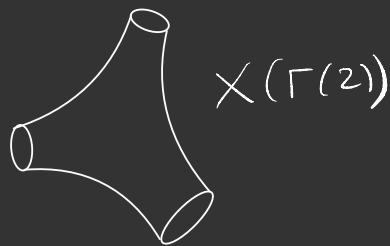
Example: $\Gamma = \Gamma(2)$

$$\varepsilon_2 = 0$$

$$\varepsilon_3 = 0$$

$$\varepsilon_\infty = 3$$

$$g = 0$$



$$\dim M_k(\Gamma(2)) = (k-1)(0-1) + \frac{k}{2} \cdot 3 = \frac{k}{2} + 1, \quad k \text{ even}$$

$$\dim M_2(\Gamma(2)) = 2$$

In the previous lectures we have shown that:

$$(\bullet) \theta_{00}^4, \theta_{10}^4, \theta_{01}^4 \in M_2(\Gamma(2))$$

$$\text{Recall: } \theta_{00}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}, \quad \theta_{01}(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 z}, \quad \theta_{10}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 z}$$

$\Rightarrow$ There is a linear relation between these $(\bullet)$ elements

$$\text{Jacobi identity: } \theta_{00}^4 = \theta_{10}^4 + \theta_{01}^4$$

Lemma: $M_2(\Gamma(2)) = \mathbb{C} \cdot \theta_{00}^4 + \mathbb{C} \cdot \theta_{01}^4$

What about other even weights?

$$M_4(\Gamma(2)) \subset \mathbb{C}[\theta_{00}^4, \theta_{01}^4]_{\deg=2}^{\text{homogeneous}} =: V_4$$

$$\dim V_4 = 3 \quad \leftarrow \text{exercise}$$

$$\dim M_4(\Gamma(2)) = 3$$

Exercise: For even k

$$M_k(\Gamma(2)) = \mathbb{C}[\theta_{00}^4, \theta_{01}^4]_{\deg=\frac{k}{2}}^{\text{homogeneous}}$$

Multiplicity vs order

$X = \mathbb{C}$ $f: \mathbb{C} \rightarrow \mathbb{P}^1$ meromorphic function $z_0 \in \mathbb{C}$

Laurent expansion of f at z_0 : $f(z) = \sum_{n=n_0}^{\infty} c_n (z-z_0)^n$,
 $c(n_0) \neq 0$

if $n_0 < 0 \Rightarrow f(z_0) = \infty$

$\text{mult}_{z_0}(f) = |n_0|$, $\text{ord}_{z_0}(f) = n_0$

if $n_0 = 0 \Rightarrow f(z_0) \in \mathbb{C} \setminus \{0\}$ $f(z) = c_0 + \sum_{n=n_1}^{\infty} c_n (z-z_0)^n$

$\text{mult}_{z_0}(f) = n_1$ $\text{ord}_{z_0}(f) = 0$

if $n_0 > 0 \Rightarrow f(z_0) = 0$ $f(z) = \sum_{n=n_0}^{\infty} c_n (z-z_0)^n$,

$\text{mult}_{z_0}(f) = n_0$ $\text{ord}_{z_0}(f) = n_0$